Hybrid approach for solving unsteady laminar forced convection inside ducts with periodically varying inlet temperature

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The transient forced convection in laminar flow inside a parallel plate channel subjected to periodically varying inlet temperature is solved by using a hybrid scheme that combines the generalized integral transform technique with a second-order accurate finite differences. Semi-analytical results are presented for the variations in the amplitude of periodically varying fluid bulk temperature and wall heat flux along the channel length for different frequencies. An approximate formula for the decay of the peak bulk temperature amplitude is developed.

Keywords: unsteady heat transfer; generalized integral transform technique; laminar flow

Introduction

Transient analysis of convective heat transfer inside ducts is important in the analysis of start-up and shutdown of heat exchange devices, or other transients arising in their operating conditions. Unsteady behavior of temperature distribution in such devices can reduce thermal performance and produce severe thermal stresses, thus jeopardizing the equipment. Therefore, accurate prediction of temperature and heat-flux distributions in unsteady forced convection are important for precise control of heat-exchange equipment. A better understanding of the behavior of regenerative and recuperative heat exchangers, gas turbines blades, and other devices under periodically operating conditions is also important. Most of the earlier studies on the subject were concerned with the simplified analysis of the problem based on slug flow approximation (Cotta and Ozisik 1986; Kakac and Yener 1973; Sparrow and de Farias 1968).

Recently, Kim, Cotta and Ozisik (1990) presented results based on the lowest-order analytical solution for transient laminar forced convection with parabolic flow inside ducts resulting from an arbitrarily varying inlet temperature by a hybrid approach using the generalized integral transform and classical Laplace transform techniques.

Kakac, Li and Cotta (1990) studied the unsteady laminar forced convection in ducts with periodic variation of inlet temperature by considering a quasi-steady response, neglecting initial transients. This approach led to a complex-valued problem that was then handled by the generalized integral transform technique.

Hatay et al. (1991) applied a second-order accurate explicit finite-differences scheme to the solution of unsteady forced convection in laminar parabolic channel flow subjected to sinusoidally varying inlet temperature: however, only the amplitudes of temperature oscillations along the centerline of the channel were reported. The stability analysis for this explicit scheme reveals that the diffusion term in the transversal coordinate introduces another restriction on the stability criteria in addition to that imposed by the Courant number. Such an undesirable effect can be circumvented by a hybrid numerical-analytical approach proposed by Cotta and Gerk (1993) where the spatial derivatives in the normal direction are removed by the application of the generalized integral transform technique, and the resulting system of hyperbolic equations in the axial direction is solved by finite differences. In addition, such a hybrid approach can provide an analytical expression for accurate computation of heat fluxes anywhere within the duct cross section, in contrast to the fully numerical approach where the accurate evaluation of gradients to compute heat fluxes require very fine meshes.

Analysis

We consider transient forced convection in thermally developing, hydrodynamically developed laminar flow between parallel plates, subjected to a sinusoidal variation of inlet temperature. We assume that the physical properties are constant, whereas viscous dissipation, free convection, and axial conduction effects are negligible. Then, the mathematical formulation of the problem is given by

$$\frac{\partial T(r,z,t)}{\partial t} + u(r)\frac{\partial T(r,z,t)}{\partial z} = \frac{\partial^2 T(r,z,t)}{\partial r^2}, \qquad 0 < r < r_w, \; z > 0, \; t > 0 \tag{1a}$$

$$T(r,z,0) = T_i, \qquad 0 \le r \le r_w, \; z \ge 0 \tag{1b}$$

$$T(r,0,t) = T_i + \Delta T \sin(\omega t), \qquad 0 \le r \le r_w, \; t > 0 \tag{1c}$$

$$\frac{\partial T(0,z,t)}{\partial r} = 0, \qquad z > 0, \; t > 0 \tag{1d}$$

$$T(r_w,z,t) = T_i, \qquad z > 0, \; t > 0 \tag{1e}$$

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This problem is now expressed in the dimensionless form

$$\frac{\partial\Theta(R,Z,\tau)}{\partial\tau}+W(R)\frac{\partial\Theta(R,Z,\tau)}{\partial Z}=\frac{\partial^2\Theta(R,Z,\tau)}{\partial R^2},$$

$$\text{in } 0<R<1,\ Z>0,\ \tau>0 \quad (2a)$$

$$\Theta(R,Z,0)=0,\qquad 0\le R\le 1,\ Z\ge 0 \quad (2b)$$

$$\Theta(R,0,\tau)=\sin(\Omega\tau),\qquad 0\le R\le 1,\ \tau>0 \quad (2c)$$

$$\frac{\partial\Theta(0,Z,\tau)}{\partial R}=0,\qquad Z>0,\ \tau>0 \quad (2d)$$

$$\Theta(1,Z,\tau)=0,\qquad Z>0,\ \tau>0 \quad (2e)$$

We proceed by seeking a formal solution to the system (Equation 2) through the combined use of the generalized integral transform technique and the finite differences method.

To remove the partial derivatives with respect to the normal variable from this system by the application of the generalized integral transform technique, we consider the following auxiliary problem:

$$\frac{d^2\Psi(\mu_i,R)}{dR^2}+\mu_i^2\Psi(\mu_i,R)=0,\qquad 0<R<1 \quad (3a)$$

$$\frac{d\Psi(\mu_i,R)}{dR}=0,\qquad R=0 \quad (3b)$$

$$\Psi(\mu_i,R)=0,\qquad R=1 \quad (3c)$$

which is readily solved to yield

$$\Psi(\mu_i,R)=\cos\mu_i R,\ i=1,2,\ldots \quad (4a)$$

$$\mu_i=\frac{(2i-1)}{2}\pi,\qquad i=1,2,\ldots. \quad (4b)$$

Using the eigenfunctions of this system, the integral transform pair with respect to the R variable is defined as

$$\textit{Inversion:}\quad \Theta(R,Z,\tau)=\sum_{i=1}^{\infty}\frac{\Psi(\mu_i,R)}{N_i^{1/2}}\bar{\Theta}_i(Z,\tau) \quad (5a)$$

$$\textit{Transform:}\quad \bar{\Theta}_i(Z,\tau)=\int_{R=0}^{1}\frac{\Psi(\mu_i,R)}{N_i^{1/2}}\Theta(R,Z,\tau)\,dR \quad (5b)$$

where N_i is the normalization integral

$$N_i=\int_{R=0}^{1}\Psi^2(\mu_i,R)dR \quad (5c)$$

Even though the integral in Equation 5c can readily be performed, for generality we prefer to carry out the analysis by using the symbol N_i.

We now operate on Equation 2a with the operator $\int_{R=0}^{1}[\Psi(\mu_i,R)/N_i^{1/2}]\,dR$ to obtain

$$\frac{\partial\bar{\Theta}_i(Z,\tau)}{\partial\tau}+\int_{R=0}^{1}W(R)\frac{\Psi(\mu_i,R)}{N_i^{1/2}}\frac{\partial\Theta(R,Z,\tau)}{\partial Z}dR+\mu_i^2\bar{\Theta}_i(Z,\tau)=0 \quad (6)$$

where the bar denotes the transform with respect to the R variable.

The integral term in Equation 6 is expressed in the alternative form by substituting the inversion formula (Equation 5a) for $\Theta(R,Z,\tau)$, to yield

$$\frac{\partial\bar{\Theta}_i(Z,\tau)}{\partial\tau}+\sum_{k=1}^{\infty}A_{ik}\frac{\partial\bar{\Theta}_k(Z,\tau)}{\partial Z}+\mu_i^2\bar{\Theta}_i(Z,\tau)=0,\ i=1,2,\ldots,$$

$$Z>0,\ \tau>0 \quad (7a)$$

where,

$$A_{ik}=A_{ki}=\frac{1}{N_i^{1/2}N_k^{1/2}}\int_{R=0}^{1}W(R)\Psi(\mu_i,R)\Psi(\mu_k,R)dR \quad (7b)$$

and the initial and inlet conditions (Equation 2b) and (Equation 2c), respectively, are transformed through the operator $\int_{R=0}^{1}[\Psi(\mu_i,R)/N_i^{1/2}]\,dR$, to give

$$\bar{\Theta}_i(Z,0)=0,\qquad Z\ge 0 \quad (7c)$$

$$\bar{\Theta}_i(0,\tau)=\bar{f}_i,\qquad i=1,2,\ldots,\ \tau>0 \quad (7d)$$

where

$$\bar{f}_i=\frac{1}{N_i^{1/2}}\int_{R=0}^{1}\sin(\Omega\tau)\Psi(\mu_i,R)dR \quad (7e)$$

The infinite system of hyperbolic equations (Equation 7a) for the transform $\bar{\Theta}(Z,\tau)$ subjected to the conditions (Equations 7c and d) can be solved by finite differences after truncating with a sufficiently large order N. Using a second-order accurate explicit finite-differences scheme based

Notation

A_{ik}	coefficients matrix, Equation 7b
D_h	hydraulic diameter ($=4\,r_w$)
N	number of terms in eigenvalue expansion
N_i	normalization integral, Equation 5c
$Q_w(Z,\tau)$	dimensionless wall heat flow
r_w	half the spacing between parallel plates, m
r	normal coordinate, m
R	dimensionless normal coordinate ($=r/r_w$)
t	time, s
T_i	inlet temperature, K
$T(r,z,t)$	fluid temperature, K
ΔT	reference temperature difference, K
u_m	mean flow velocity, m/s
$u(r)$	flow velocity, m/s
$W(R)$	dimensionless flow velocity ($=u(r)/16\,u_m=3(1-R^2)/32$)
y	discretization parameter in the normal direction ($=\Delta\tau/(\Delta R)^2$)
z	axial coordinate, m
Z	dimensionless axial coordinate ($=\alpha z/u_m D_h^2$)

Greek letters

α	thermal diffusivity of the fluid, m^2/s
γ	Courant number ($=c\,\Delta\tau/\Delta Z$)
λ	discretization parameter, defined by Equation 8d
$\Theta(R,z,\tau)$	dimensionless temperature ($=[T(r,z,t)-T_i/\Delta T]$)
$\Theta_b(Z,\tau)$	dimensionless fluid bulk temperature, defined by Equation 9
$\Theta_{bm}(Z)$	dimensionless peak bulk temperature
$\Psi(\mu_i,R)$	eigenfunctions of eigenvalue problem 3
μ_i	eigenvalues of eigenvalue problem 3
τ	dimensionless time ($=\alpha t/r_w^2$)
Ω	dimensionless frequency of oscillations ($=\omega r_w^2/\alpha$)
ω	frequency of oscillations, Hz

on an extension of the Warming and Beam (1976) upwind scheme, the system (Equation 7) becomes

Predictor:

$$\Theta_{i,j}^{\overline{n+1}} = \Theta_{i,j}^{n} - \lambda \sum_{k=1}^{N} A_{ik}(\Theta_{k,j}^{n} - \Theta_{k,j-1}^{n}) - \mu_i^2 \Delta\tau \Theta_{i,j}^{n} \tag{8a}$$

Corrector:

$$\Theta_{i,j}^{n+1} = \frac{1}{2}\Big[\Theta_{i,j}^{n} + \Theta_{i,j}^{\overline{n+1}} - \lambda \sum_{k=1}^{N} A_{ik}(\Theta_{k,j}^{\overline{n+1}} - \Theta_{k,j-1}^{\overline{n+1}}) - \lambda \sum_{k=1}^{N} A_{ik}(\Theta_{k,j}^{n} - 2\Theta_{k,j-1}^{n} + \Theta_{k,j-2}^{n}) - \mu_i^2 \Delta\tau \Theta_{i,j}^{\overline{n+1}}\Big] \tag{8b}$$

where,

$$\lambda = \frac{\Delta\tau}{\Delta Z} \quad \text{and} \quad \Theta_{i,j}^{n} \equiv \Theta(n\Delta\tau, j\Delta Z) \tag{8c, d}$$

and the superscript $\overline{n+1}$ denotes evaluation at an intermediate time.

To solve this system one needs to march along Z and τ up to a final axial position and time. Then, the unknown temperature $\Theta(R, Z, \tau)$ is recovered at any desired normal position R, for the axial positions Z and time τ specified in the numerical solution, by the application of the inversion formula (Equation 5a).

The stability analysis of this scheme, presented in the appendix, leads to the following stability criteria:

$$0 \le \gamma_i \le 2, \qquad i = 1, 2, \ldots, N \tag{8e}$$

where $\gamma_i = c_i \Delta\tau/\Delta Z$ is the Courant number.

Table 1 Comparison of present hybrid solution Θ_{hyb} with the lowest-order analytical solution Θ_{los} given by Kim et al. (1990) and the numerical solution Θ_{num} given by Hatay et al. (1991), for $\Omega = 0.05$, 0.1 and 0.5 ($\gamma = 0.625$, $Y = 0.18$)

τ	Z	5×10^{-3}	1×10^{-2}	2×10^{-2}	3×10^{-2}	5×10^{-2}	1×10^{-1}
				$\Omega = 0.05$			
1	Θ los	0.0372	0.0296	0.0188	0.0117	0.0039	0
	Θ hyb	0.0397	0.0338	0.0249	0.0184	0.0101	0
	Θ num	0.0399	0.0340	0.0251	0.0185	0.0101	0
5	Θ los	0.1937	0.1630	0.1173	0.0846	0.0439	0.0084
	Θ hyb	0.1986	0.1689	0.1247	0.0922	0.0504	0.0112
	Θ num	0.2043	0.1731	0.1273	0.0943	0.0516	0.0114
10	Θ los	0.3779	0.3200	0.2334	0.1707	0.0912	0.0190
	Θ hyb	0.3850	0.3275	0.2416	0.1787	0.0977	0.0216
	Θ num	0.3950	0.3345	0.2466	0.1824	0.0998	0.0211
				$\Omega = 0.10$			
1	Θ los	0.0743	0.0592	0.0376	0.0234	0.0079	0
	Θ hyb	0.0793	0.0675	0.0498	0.0369	0.0202	0
	Θ num	0.0797	0.0678	0.0500	0.0370	0.0203	0
5	Θ los	0.3757	0.3164	0.2281	0.1647	0.0857	0.0165
	Θ hyb	0.3849	0.3274	0.2415	0.1786	0.0977	0.0216
	Θ num	0.3950	0.3345	0.2466	0.1824	0.0998	0.0221
10	Θ los	0.6643	0.5635	0.4124	0.3024	0.1626	0.0343
	Θ hyb	0.6758	0.5748	0.4239	0.3135	0.1715	0.0379
	Θ num	0.6933	0.5871	0.4329	0.3202	0.1752	0.0388
				$\Omega = 0.50$			
1	Θ los	0.3586	0.2870	0.1839	0.1149	0.0391	0
	Θ hyb	0.3810	0.3244	0.2393	0.1770	0.0968	0
	Θ num	0.3827	0.3257	0.2403	0.1777	0.0972	0
5	Θ los	0.4935	0.4367	0.3443	0.2704	0.1634	0.0423
	Θ hyb	0.4805	0.4087	0.3014	0.2229	0.1219	0.0270
	Θ num	0.4931	0.4175	0.3079	0.2277	0.1246	0.0276
10	Θ los	−0.7665	−0.6578	−0.4912	−0.3663	−0.2015	−0.0423
	Θ hyb	−0.7701	−0.6550	−0.4831	−0.3573	−0.1954	−0.0432
	Θ num	−0.7901	−0.6690	−0.4933	−0.3649	−0.1996	−0.0442

Knowing the dimensionless temperature profile $\Theta(R, Z, \tau)$, the fluid bulk temperature $\Theta_b(Z, \tau)$ can be evaluated from its definition

$$\Theta_b(Z, \tau) = 16 \int_0^1 W(R)\Theta(R, Z, \tau)dR \tag{9}$$

An alternative form of this expression, which is more convenient for computational purposes, is obtained by replacing $\Theta(R, Z, \tau)$ by its equivalent inversion formula (Equation 5a) as

$$\Theta_b(Z, \tau) = 16 \sum_{i=1}^{N} \frac{\bar{g}_i}{N_i^{1/2}} \bar{\Theta}_i(Z, \tau) \tag{10a}$$

where,

$$\bar{g}_i = \int_{R=0}^{1} W(R)\Psi(\mu_i, R)dR \tag{10b}$$

Also of interest is the evaluation of the dimensionless heat flux at the outer boundary, $\partial\Theta(1, Z, \tau)/\partial R$, for which an explicit analytical expression is readily obtained according to the inversion formula (Equation 5a) as

$$\frac{\partial\Theta(1, Z, \tau)}{\partial R} = \frac{1}{N_i^{1/2}} \sum_{i=1}^{N} \frac{d\Psi(\mu_i, 1)}{dR} \bar{\Theta}_i(Z, \tau) \tag{11}$$

Results and discussion

We now present numerical results for the fluid bulk temperature and wall heat flux, and compare the present hybrid method of solution to the lowest-order analytical and numerical methods of solution reported in the literature. We made the comparison for the case of three dimensionless frequencies, Ω = 0.05, 0.1 and 0.5.

Table 1 shows a comparison of fluid bulk temperature distributions given by the present hybrid approach, the purely numerical solution by Hatay et al. (1991) and the lowest-order analytical solution by Kim et al. (1990). The accuracy of the lowest-order solution decreases for small dimensionless times and large distances from the inlet. The agreement between the purely numerical and hybrid approaches is very good; the maximum deviation being 2.6 percent for large times near the inlet. However, the limitation to the accuracy of the purely numerical solution technique for the calculation of heat flux, resulting from the stability restriction imposed on the size of normal steps, should be recognized. Because the evaluation of heat flux distributions depends on numerical computation of derivatives with respect to the normal variable, the normal mesh size, ΔR, has to be small enough to ensure accuracy, especially near the channel inlet where gradients are steeper. Conversely, with the hybrid approach, heat-flux distributions at any specified location within the duct cross section can be evaluated a posteriori, using the explicit analytic expression given by the derivative of the inversion formula (Equation 5a).

For the situation shown on Table 1, the execution times for both schemes is practically the same; however, the execution time can be further reduced and dispersive errors virtually eliminated with the hybrid scheme by increasing the Courant number toward unity. Such flexibility is limited when dealing with the purely numerical approach because an increase in the Courant number requires a decrease in the value of the parameter $y = \Delta\tau/(\Delta R)^2$ to satisfy the more restricted stability criteria. Because ΔR has to remain small, a decrease in the parameter y requires a decrease in $\Delta\tau$ as well. The stability of the present hybrid approach depends only on the value of the Courant number, whereas in the purely numerical scheme two different stability criteria need to be satisfied.

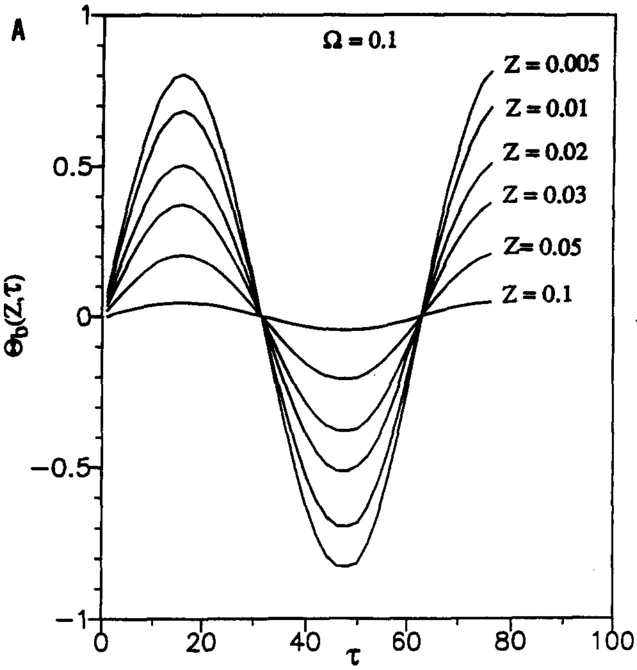

Figure 1a Variation of the bulk temperature as a function of time for $\Omega = 0.1$

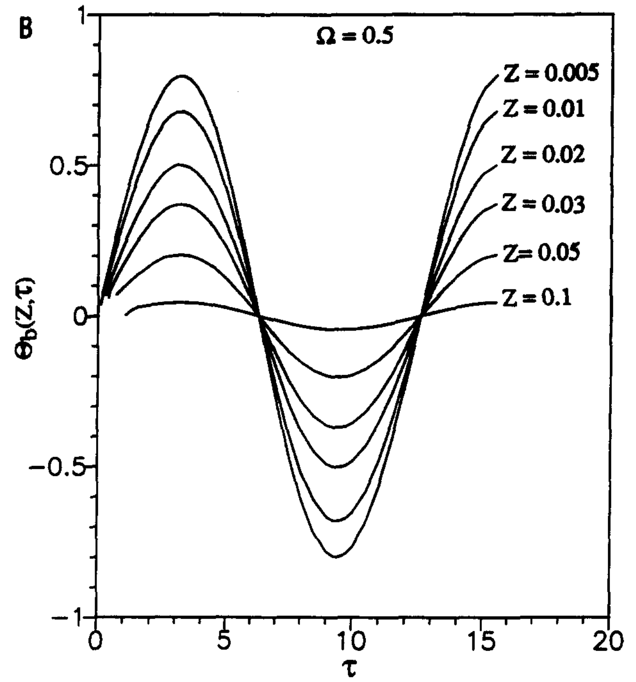

Figure 1b Variation of the bulk temperature as a function of time for $\Omega = 0.5$

The present results with the hybrid method were fully converged with a maximum of N = 15 terms in the series, and the Courant number value was maintained as 0.93.

Figures 1a and b show the variation of the bulk temperature as a function of time at different axial positions for $\Omega = 0.1$ and 0.5, respectively, for sinusoidal oscillation of inlet temperature. As expected, the amplitude becomes smaller with increasing distance from the inlet. This trend is better envisioned in Figure 2, which shows the maximum (peak) bulk temperature decaying exponentially with the axial distance from the inlet. The effects of frequencies within the range

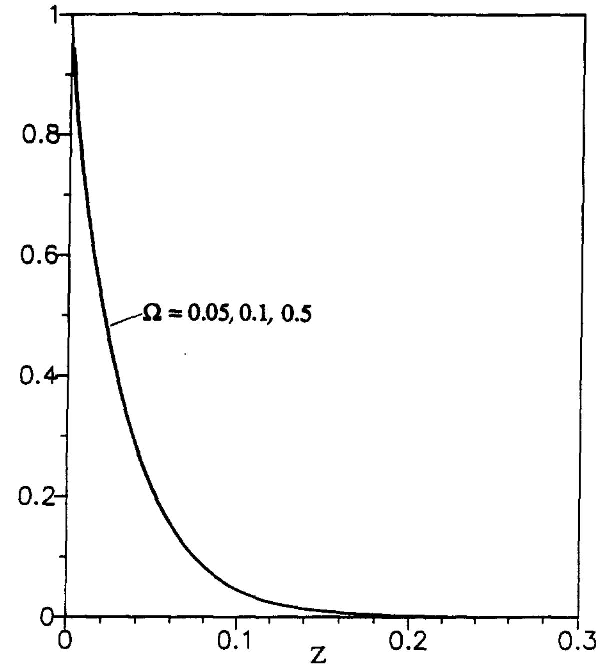

Figure 2 Variation of the maximum (peak) bulk temperature as a function of the axial locations

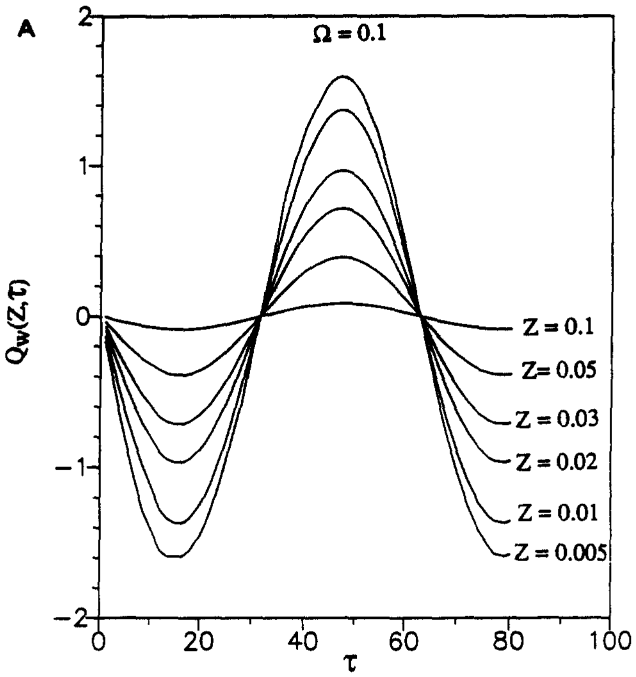

Figure 3a Variation of the wall heat flux as a function of time for $\Omega = 0.1$

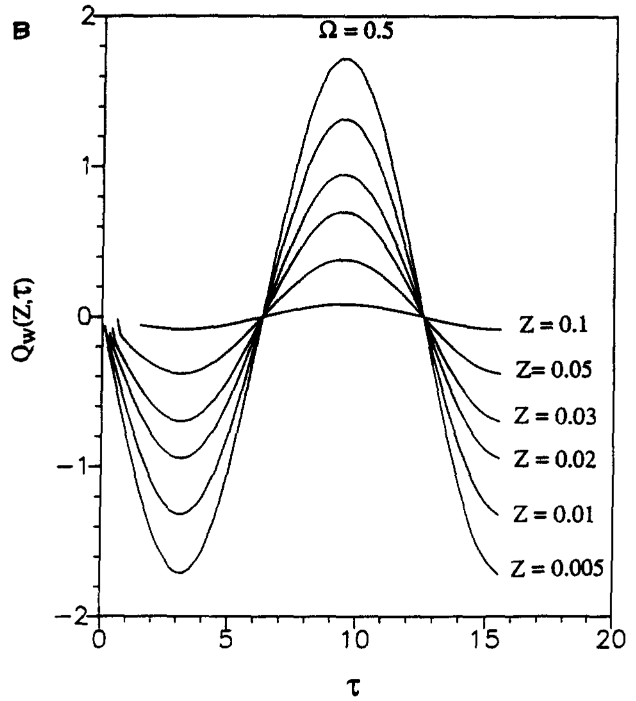

Figure 3b Variation of the wall heat flux as a function of time for $\Omega = 0.5$

considered here on the peak bulk temperature appear to be negligible.

Figures 3a and b show the variation of the heat flux at the outer boundary of the channel as a function of time at different axial positions for frequencies $\Omega = 0.1$ and 0.5, respectively. A comparison of the results in Figures 1 and 3 reveals that when the fluid bulk temperature is greater than the wall temperature, the heat flux is negative, which implies that the heat flows from the fluid to the wall and vice versa.

Based on the cases studied here, it seems that the peak bulk temperature varies exponentially with the dimensionless axial distance Z along the channel. Therefore, we developed the following approximate formula for the decay of the peak bulk temperature with distance

$$\Theta_{bm}(Z) = \exp[-(aZ + b)] \quad (12)$$

where the coefficients a and b are listed on Table 2.

Table 2 Coefficients a and b of Equation 12

Range of Z	a	b	Max error in Θ_{bm} (percent)
$Z \leq 0.006$	40.855	0.0262	1.6
$Z > 0.006$	30.175	0.0892	2.0

Conclusions

A hybrid numerical-analytical methodology is presented for solving laminar heat transfer in the thermally developing region of a parallel plate channel subjected to periodic variation of inlet temperature. The method has advantages over conventional purely numerical approaches in that the heat flux anywhere in the medium can be computed a posteriori by using analytical expressions. Also, the method is less restricted by stability considerations, thus allowing more flexibility to improve the accuracy of computations. The results show that the amplitude of oscillations of bulk temperature and wall heat flux decays with the distance along the duct. An analytical expression is presented for the calculation of the exponential decay of the peak bulk temperature.

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Appendix: Stability analysis

To examine the stability of the explicit numerical scheme used here for the solution of system (Equation 7), we rewrite this system in the form of the classical wave equation, through the transformation

$$\bar{\Theta}_i^*(Z,\tau) = \bar{\Theta}_i(Z,\tau)\exp\left[\mu_i^2\tau\right] \tag{A1}$$

which yields the alternative form

$$\frac{\partial \mathbf{U}}{\partial \tau} + \mathbf{A}^*(\tau)\frac{\partial \mathbf{U}}{\partial Z} = 0 \tag{A2}$$

where,

$$\mathbf{A}^*(\tau) = \{A_{ik}^*(\tau)\} = A_{ik}\exp[-\mu_k^2 - \mu_i^2)\tau] \tag{A3}$$

with A_{ik} given by Equation 7b,

$$\mathbf{U} = \{\bar{\Theta}_1^*(Z,\tau), \bar{\Theta}_2^*(Z,\tau), \ldots, \bar{\Theta}_N^*(Z,\tau)\}^T \tag{A4}$$

and the superscript T denotes the transpose.

Warming and Beam (1976) studied the stability of the upwind scheme applied to a model equation similar to Equation A2. In their analysis, the coefficients matrix $\mathbf{A}^*$ was assumed constant to apply linear stability theory. Then, by applying a von Neumann type stability analysis, it can be shown that the upwind scheme is stable if and only if

$$0 \le \gamma_i \le 2, \qquad i = 1, 2, \ldots, N \tag{A5}$$

where the Courant number γ_i is defined as

$$\gamma_i = c_i\frac{\Delta\tau}{\Delta Z} \tag{A6}$$

for all eigenvalues c_i of $\mathbf{A}^*$ and with the restriction that $c_i > 0$.